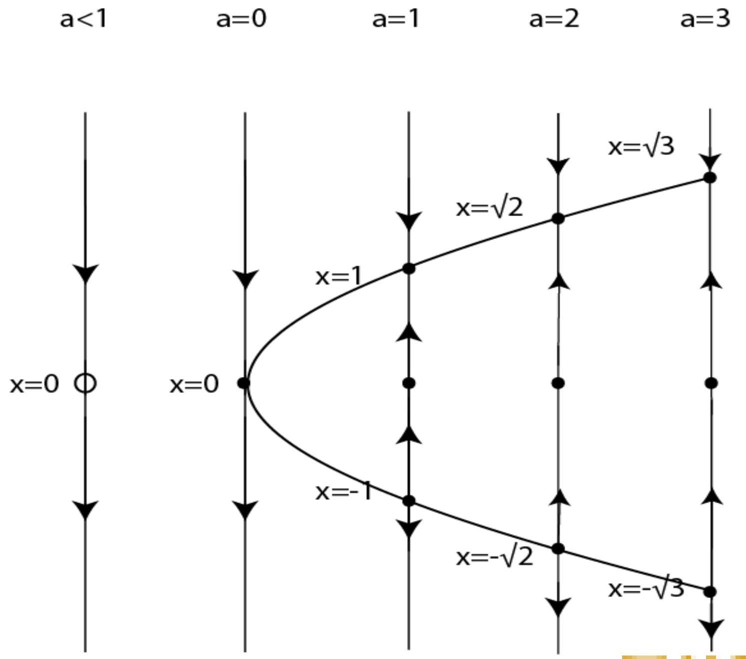
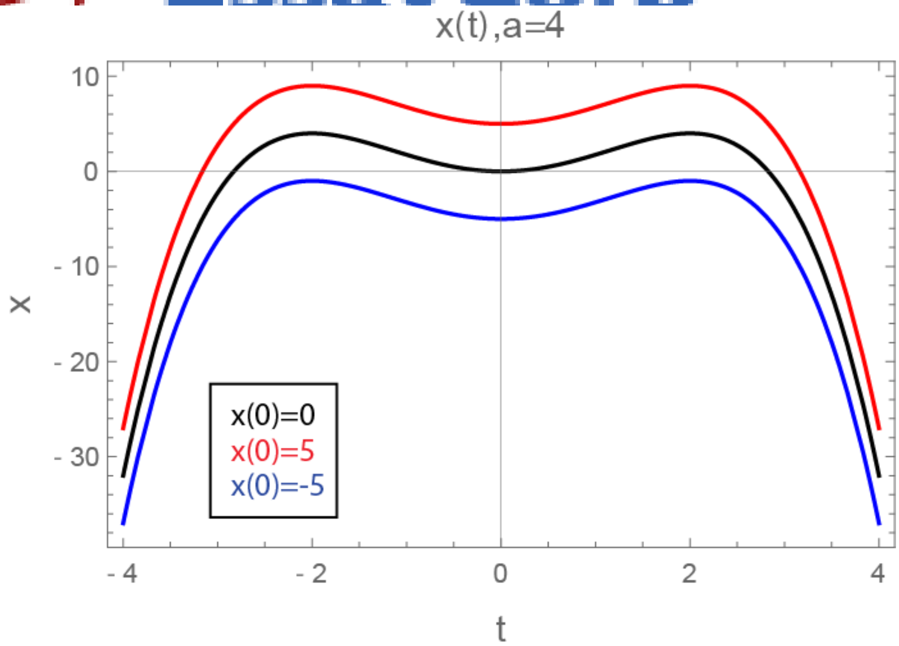
1. a) The fixed points exist where $\frac{dx}{dt} = 0$, that is, for $x = 0$ (unstable) and $x = \pm\sqrt{a}$ (stable). The fixed point at $x = \pm\sqrt{a}$ exists only if $a \geq 0$ (for real solutions).

   b)

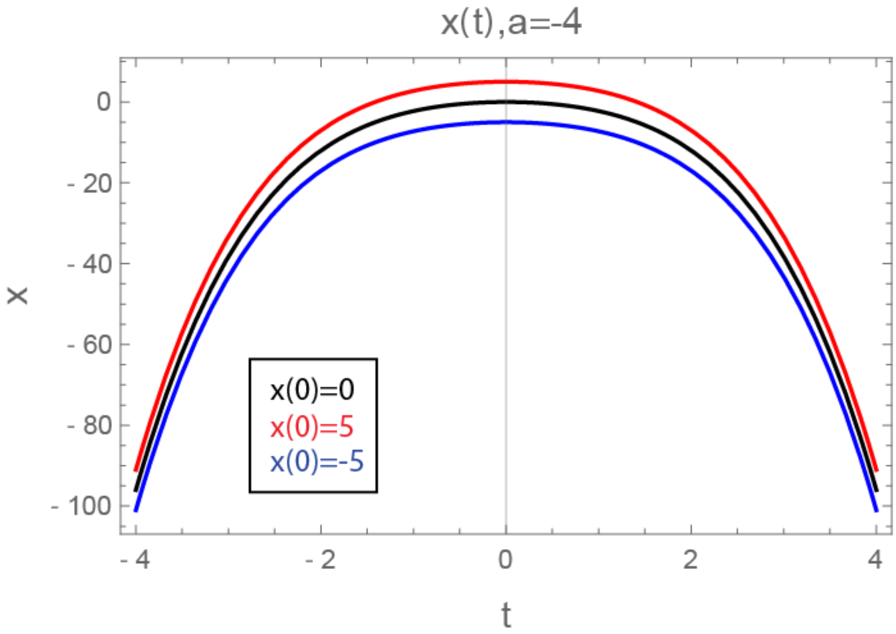



c) $x(t) = \frac{a\,x^2}{2} - \frac{x^4}{4} + x(0)$, so for a=4 we have the following evolution






Where we can see 1 stable fixed point at x=0 and 2 unstable fixed points at $x = \pm\sqrt{2}$. The evolution for n=-4 is shown at the following graph. The only fixed point is at x=0, as expected.


x(t),a=-4
0
- 20
- 40
- 60
- 80
- 100
x
- 4
- 2
0
2
4
t
x(0)=0
x(0)=5
x(0)=-5

2) a) For fixed points to exist, both equations should be equal to zero. That gives

$$\omega = -\frac{g}{l} Sin\theta = 0$$

That gives fixed points for

$$\omega = n\pi = 0$$

Which are of two kinds, each for even or odd value of n. For n odd, θ=π (or equivalently 3π,5π etc.), which is an unstable (hyperbolic) fixed point, and for n even, θ=0 (2π,4π etc.) which is a stable (elliptic) fixed point.

b) For n even, the position of the pendulum is the vertical downward, and it does not move. A small perturbation (a push) introduced will produce a small oscillating motion around its initial position, with a frequency ω.

For n odd, the pendulum is at the vertical upward position, opposite to the one for n even. It will remain at rest for as much as it is perfectly stable at this position, but a very small perturbation will be enough to make it fall from this position towards the downwards vertical. After this, it will oscillate around the stable fixed point.

c) For H to be a constant of the system, it should be

$$\frac{dH}{dt} = 0$$

$$\Rightarrow \frac{dH}{d\theta}\frac{d\theta}{dt} + \frac{dH}{d\omega}\frac{d\omega}{dt} = 2\omega g Sin\theta + 2l\omega * \left(-\frac{g}{l} Sin\theta\right) = 0$$

Thus H(t) is a constant of the motion.

d) For H>C, the pendulum will swing around and around, passing through all possible angles(rotation). For H<C, the pendulum will only swing around the equilibrium position, which is the downward vertical position (θ=0) (libration)

For H=C, the motion of the pendulum depends on its state at the time. If it is already in motion, it will always approach the vertical position, without reaching it. If at rest in a vertical position, it will stay at this position indefinitely.

This critical value is the value of the Hamiltonian at the fixed points, that is, at the vertical positions ($\theta = \pm\pi$). There, the Hamiltonian becomes

$$c = E_{crit} = 2mgL$$

3. a) A particular solution of the damped harmonic oscillator should be

$$x(t) = R\ Sin(\omega t + \varphi)$$

Where R is the response of the oscillator and ϕ a phase. Introducing this solution to the equation, we get

$$-R\ \omega^2\ Sin(\omega t + \varphi) - \delta\ R\ \omega\ Cos(\omega t + \varphi) + R\ \kappa^2\ Sin(\omega t + \varphi) = A\ Sin\omega t$$

Using the identities

$$Sin(\omega t - \varphi) = Sin\omega t\ Cos\varphi - \boldsymbol{Cos\omega t\ Sin\varphi}$$

$$Cos(\omega t - \varphi) = Cos\omega t\ Cos\varphi + Sin\boldsymbol{\omega t\ Sin\varphi}$$

And making the coefficients of Cosωt and Sinωt separately equal to zero,

$$R(\omega^2 - \kappa^2)Cos\varphi + R\delta\kappa\ Sin\varphi - A = 0$$

$$(\omega^2 - \kappa^2)Sin\varphi - \delta\kappa\ Cos\varphi = 0$$

The latter equation gives

$$Tan\varphi = \frac{\delta\kappa}{(\omega^2 - \kappa^2)}$$

And so



$$Sin\varphi = \frac{1}{\sqrt{1 + Tan^2\varphi}} = \frac{\omega^2 - \kappa^2}{\sqrt{(\omega^2 - \kappa^2)^2 + \delta^2\kappa^2}}$$

$$Cos\varphi = \frac{Tan\varphi}{\sqrt{1 + Tan^2\varphi}} = \frac{\delta\ \kappa}{\sqrt{(\omega^2 - \kappa^2)^2 + \delta^2\kappa^2}}$$

Replacing these equations for Cosϕ and Sinϕ into the coefficients of Cosωt and Sinωt,

we get an equation for R

$$R(\omega) = \frac{A}{\sqrt{(\kappa^2 - \omega^2)^2 + \delta^2\omega^2}}$$

b) The response is larger when κ=ω, i.e. the forcing frequency is equal to the frequency of the system. Mathematically, for this forcing frequency, the denominator becomes the smallest possible and R(ω) takes its largest value. Ω=κ is called the resonance frequency.

4. a) The equations of displacement for this system are generally:

$$m_i \frac{d^2\xi_i}{dt^2} = -k_i\, \xi_{i-1} - (k_i + k_{i+1})\xi_i + k_{i+1}\xi_{i+1}\,, i = 2, \ldots, n-1$$

Which only serve for n=2 for said system as they are. For the other two cases, we have

$$m_i \frac{d^2\xi_i}{dt^2} = -(k_i + k_{i+1})\xi_i + k_{i+1}\xi_{i+1}\,, i = 1$$

$$m_i \frac{d^2\xi_i}{dt^2} = -k_i\, \xi_{i-1} - (k_i + k_{i+1})\xi_i\,, i = 3$$

Since $k_i = 1$ for all springs and $m_2 = M, m_{1,3} = 1$, we get

$$\frac{d^2x_1}{dt^2} = -x1 + x_2$$

$$M\frac{d^2x_2}{dt^2} = x_3 - 2x_2 + x_1$$

$$\frac{d^2x_3}{dt^2} = -x_3 + x_2$$

b) The solutions of the displacement equations are $\xi_i(t) = u_i e^{i\omega t}$. Substituting these solutions to the equations derived in a), we get the following system:



$$\begin{pmatrix} \omega^2 - 1 & 1 & 0 \\ 1 & M\omega^2 - 2 & 1 \\ 0 & 1 & \omega^2 - 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

The eigenvalues of which are derived by setting the determinant equal to 0. This way, we obtain the equation

$$(\omega^2 - 1)[(M\omega^2 - 2)(\omega^2 - 1) - 2] = 0$$

The solutions of which are $\omega = 0, \omega = 1\ and\ \omega^2 = 1 + \frac{2}{M}$

c) Plugging the eigenvalues for $\omega$ in the system, we get

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \omega = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & M-2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \omega = 1$$

$$\begin{pmatrix} \frac{2}{M} & 1 & 0 \\ 1 & M & 1 \\ 0 & 1 & \frac{2}{M} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \omega = 1 + \frac{2}{M}$$

For ω=0, the first and the second row give $u_1 = u_2 = u_3$, but since there is no frequency , the bodies are at rest. For ω=1, the solutions are $u_1 = -u_3, u_2 = 0$. This means that the body B does not move, but the other two oscillate around it, in opposite directions every time. Finally, for $\omega = 1 + \frac{2}{M}$,

$$(1 - \frac{2}{M})u_1 = u_3$$

$$u_2 = -\frac{2}{M}(\frac{1 - \frac{1}{M}}{1 - \frac{2}{M}})\, u_3$$

Where all bodies oscillate, but the direction of the oscillation depends on the mass M of the second body.

- If $M < 1, u_1 and\ u_3$oscillate in the opposite directions, and $u_2$oscillates in the same direction with $u_1$.
- If $1 > M > 2, u_1 and\ u_3$oscillate in the opposite directions, and $u_2$oscillates in the same direction with $u_3$.
- If $M > 2, u_1 and\ u_3$oscillate in the same directions, and $u_2$oscillates in the opposite direction with both of them.

5. a) We employ separation of variables x and y for the wave equation:

$$u(x,t) = X(x)T(t)$$

So the wave equation becomes

$$c^2 X''(x)T(t) = X(x)T\ddot{(}t) \Rightarrow$$

$$-\frac{X''(x)}{X(x)} = -\frac{T\ddot{(}t)}{T(t)c^2} = \lambda, with\ \lambda > 0$$

$$\Rightarrow -X''(x) = \lambda\, X\,(x)$$

And $$-\ddot{T}(t) = \lambda c^2 T(t)$$

Since $\lambda > 0,\ X''(x) + \lambda^2\, X\,(x) = 0$

$$\Rightarrow X(x) = A\ Sin\ (ax) + B\ Cos(bx)$$

And since $X(0) = X(L) = 0$,

B=0 and

$$X(L) = A\ Sin\ (a\ L) \Rightarrow$$

$$a = n\ \pi/L$$

Thus $$X_n(x) = A_n\ Sin\ \left(\frac{n\pi}{L}x\right) \Rightarrow X(x) = \sum_{n=1}^{\infty} A_n\ Sin\ \left(\frac{n\pi}{L}x\right)$$

Since there are infinite eigenvalues, one for each n.

Working the same way for T(t), we obtain

$$T_n(t) = A_n\ Sin\ \left(\frac{n\pi}{L}ct\right) + B_n Cos\ \left(\frac{n\pi}{L}ct\right)$$

Where, applying the boundary conditions

$$u(x,0) = 0$$

We obtain $T_n(t) = C_n\ Sin\ \left(\frac{n\pi}{L}ct\right) \Rightarrow T(t) = \sum_{n=1}^{\infty} C_n\ Sin\ \left(\frac{n\pi}{L}ct\right)$

Returning to the separation of variables,

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} A_n\ C_n\ Sin\ \left(\frac{n\pi}{L}ct\right)\ Sin\ \left(\frac{n\pi}{L}x\right) \Rightarrow$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n\ \ Sin\ \left(\frac{n\pi}{L}x\right) Sin\ \left(\frac{n\pi}{L}ct\right) \Rightarrow$$

b) $\quad a_n\ = \frac{2}{n\pi c}\int_0^L Sin\ (\frac{n\pi}{L}x)\mathrm{V(x)dx}$

c) $\quad$ For $\mathrm{V(x)} = \mathrm{Sin}4\pi x$, $a_n = 2\frac{1-Cos(n\pi)}{n^2\pi^2 c} = \begin{cases} 0, n\ even \\ \frac{4}{n^2\pi^2 c}, n\ odd \end{cases}$

6. a) The volume of an ellipsoid should be given by

$$V = \int A(x)\, dx$$

Where A(x) is the surface of a 2D ellipsoid.

$$\frac{y^2 + z^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2 + z^2}{b^2(1 - \frac{x^2}{a^2})} = 1$$

Which is a circle with a radius of

$R = b\sqrt{1 - \frac{x^2}{a^2}},$

so its surface is

$$A(x) = \pi b^2 (1 - \frac{x^2}{a^2})$$

So

$$V = \int_{-a}^{a} \pi b^2 (1 - \frac{x^2}{a^2})\, dx$$

$$= \pi b^2 \int_{-a}^{a} (1 - \frac{x^2}{a^2})\, dx = \pi b^2 \left(x - \frac{x^3}{3a^2}\right)\Big|_{-a}^{a} = \pi b^2 \left[\left(a - \frac{a}{3}\right) - \left(-a + \frac{a}{3}\right)\right] =$$

$$= \frac{4}{3}\pi a b^2$$

b) The volume of a rectangle is 2x2y2z=8xyz.

We will use the identity

$$x_1 x_2 x_3 \leq (\frac{x_1 + x_2 + x_3}{3})^3$$

Where $x_1 = \frac{x^2}{a^2}, x_2 = \frac{y^2}{b^2}, x_3 = \frac{z^2}{b^2}$

But $x_1 + x_2 + x_3 = 1$,the equation of the ellipsoid, so the inequality becomes

$$\frac{x^2 y^2 z^2}{a^2 b^4} \leq \frac{1}{27}$$

And using V=8xyz,

$$\frac{V^2}{64\, a^2 b^4} \leq \frac{1}{27}$$

So the maximum volume is

$$V_{\ rect\ max} = \frac{8\ a\ b^2}{3\sqrt{3}}$$

And

$$\frac{V_{\ rect\ max}}{V_{\ ell}} = \frac{8\ a\ b^2}{3\sqrt{3}} / \frac{4\ \pi\ a\ b^2}{3} \Rightarrow$$

$$\frac{V_{\ rect\ max}}{V_{\ ell}} = 0.367{\sim}37\%$$

7.a) A slack variable is introduced to a “less than or equal” inequality of a linear program, and transforms it to an equality. It represents the slack quantity of resources, that is, the amount of unused resources. A surplus variable is introduced to a “more than or equal” inequality to transform it to an equality, and represents the amount produced in excess of the minimum requirement of the program.

b) Maximize $z = 5x + 4y$

subject to

$$3x + 4y + s_1 = 40$$

$$2x + y + s_2 = 20$$

$$y + s_3 = 6$$

$$x, y, s_1, s_2, s_3 \geq 0$$

Where $s_1, s_2$ slack variables and $s_3$ a surplus variable.