1.

(a)

Let $A=\{1,3,5,7,9\}$ and $B=\{1,4,7,8,9,10\}$ be subsets of the universal set $\xi=\{x\in\mathbf{Z}:1\le x\le 10\}$

(i)

Find $A\cap B$.

$$\begin{aligned}A\cap B&=\{1,3,5,7,9\}\cap\{1,4,7,8,9,10\}\\&=\{1,7,9\}\end{aligned}$$

(ii)

Find $A\cup B$.

$$\begin{aligned}A\cup B&=\{1,3,5,7,9\}\cup\{1,4,7,8,9,10\}\\&=\{1,3,5,7,9,4,8,10\}\end{aligned}$$

(iii)

Find $A-B$.

$$\begin{aligned}A-B&=\{1,3,5,7,9\}-\{1,4,7,8,9,10\}\\&=\{3,5\}\end{aligned}$$

(iv)

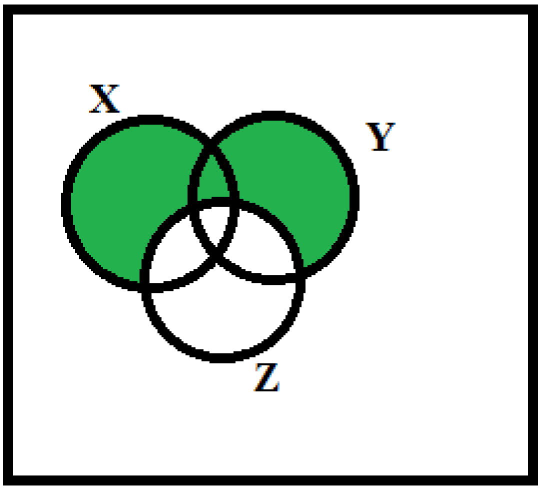
Find A' .

$$\begin{aligned}&A'=\xi-A\\&=\{1,2,3,4,5,6,7,8,9,10\}-\{1,3,5,7,9\}\\&=\{2,4,6,8,10\}\end{aligned}$$

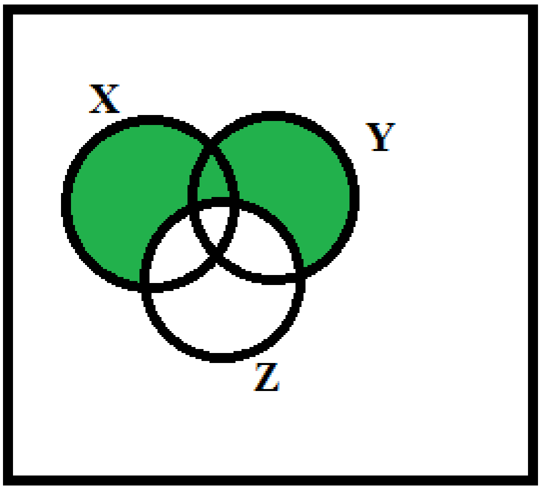
(b)

Let $X, Y,$ and Z be sets.

Draw the Venn diagram of $(X\cup Y)\cap Z'$.



Draw the Venn diagram of $X \cup (Y \cap Z')$.



From the above Venn diagrams.

$$(X \cup Y) \cap Z' = X \cup (Y \cap Z')$$

(c)

Let $P, Q,$ and R be finite sets.

And $|P| = 58, |Q| = 71, |R| = 40, |P \cap Q| = 21, |P \cap R| = 11, |Q \cap R| = 13$ and $|P \cap Q \cap R| = 7$.

Find $|P \cup Q \cup R| = ?$

Use the following formula.

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$\begin{aligned}|P \cup Q \cup R| &= |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R| \\ &= 58 + 71 + 40 - 21 - 11 - 13 + 7 \\ &= 131\end{aligned}$$

(d)

(i)

Let R and T be sets and $|R| = m$ and $|T| = n$.

$$\begin{aligned}|P(R) \times P(T)| &= |P(R)| \cdot |P(T)| \\ &= 2^m \cdot 2^n \qquad As\, |P(R)| = 2^m, |P(T)| = 2^n \\ &= 2^{m+n}\end{aligned}$$

$$\begin{aligned}|P(R \times T)| &= |P(R \times T)| \\ &= 2^{mn}\end{aligned}$$

As the set $R \times T$ has mn elements.

Set both equal and determine the values of m and n.

$$2^{m+n} = 2^{mn}$$

$$m + n = mn$$

The possible solutions are:

$m = 0, n = 0$ and $m = 2, n = 2$.

(ii)

Suppose $R = \{a, b\}$ and $T = \{1, 2, 3\}$.

(I)

$$R \times T = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

$$T^2 = T \times T = \left\{ \begin{matrix} (1,1),(1,2),(1,3), \\ (2,1),(2,2),(2,3), \\ (3,1),(3,2),(3,3) \end{matrix} \right\}$$

2.

(a)

Let $\alpha : \mathrm{R} \to \mathrm{R}^+$ and $\beta : \mathrm{R}^+ \to \mathrm{R}$ defined by:

$$\alpha(x) = x^2 - 3x + 7 \text{ and } \beta(x) = \frac{2-x}{x+1}$$

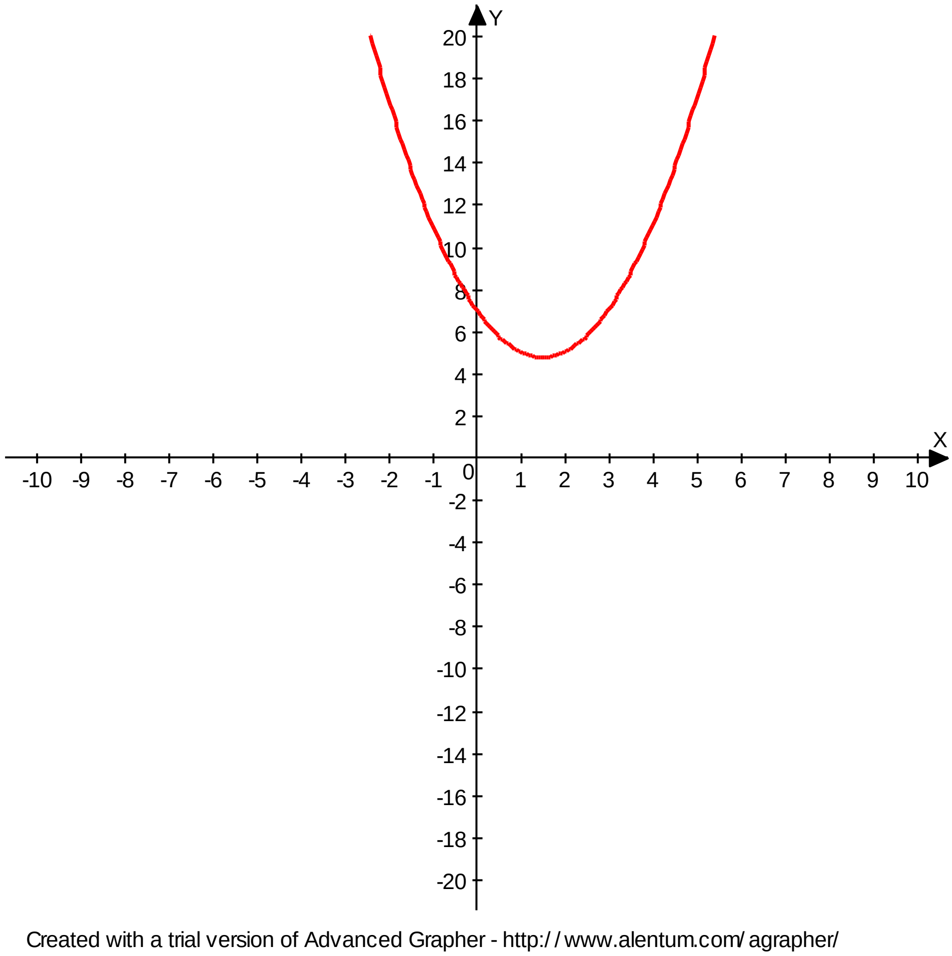
Show that α is neither injective nor surjective.

$$\alpha(0) = 0^2 - 3(0) + 7 = 7$$
$$\alpha(3) = 3^2 - 3(3) + 7 = 7$$

But, $0 \neq 3$. Thus, α is not injective.

Next, let $y = 1$. There is no $x \in \mathrm{R}$ such that $x^2 - 3x + 7 = 1$. Therefore, α is not surjective.

The graph of the function is shown below:



(b)

Show that $\beta(x)=\dfrac{2-x}{x+1}$ is injective.

Suppose there exist $a,b\in\mathbb{R}^{+}$ such that $\beta(a)=\beta(b)$.

$$\begin{aligned}\frac{2-a}{a+1}&=\frac{2-b}{b+1}\\(2-a)(b+1)&=(2-b)(a+1)\\2b+2-ab-a&=2a+2-ab-b\\2b&=2a\\a&=b\end{aligned}$$

Thus, $\beta(x)=\dfrac{2-x}{x+1}$ is injective function.

(c)

Find the range of the function $\beta(x) = \frac{2-x}{x+1}$.

The range of the function is $\{y \in R \mid y = -1\}$

(d)

$\beta^X = \frac{2-x}{x+1}$

let $\beta^X(x) = y$

$\frac{2-x}{x+1} = y$

$2 - x = y(x+1)$

$2 - y = x + xy$

$x = \frac{2-y}{1+y}$

now $\beta^X(x) = y \Rightarrow x = (\beta^X)^{-1}(y)$

$so \quad x = \frac{2-y}{1+y} \Rightarrow$

$(\beta^X)^{-1}(y) = \frac{2-y}{1+y}$

replace y by x

$(\beta^X)^{-1}(x) = \frac{2-x}{1+x}$

Therefore, the inverse of (β^X) is $\frac{2-x}{1+x}$.

(e)

Find $\alpha o \beta$.

$$\alpha o\beta(x) = \alpha(\beta(x))$$
$$= \alpha\left(\frac{2-x}{x+1}\right)$$
$$= \alpha\left(\frac{2-x}{x+1}\right)$$
$$= \left(\frac{2-x}{x+1}\right)^2 - 3\left(\frac{2-x}{x+1}\right) + 7$$
$$= (11x^2 + 7x + 5)/(x+1)^2$$

Find $\beta o\alpha$.

$$\beta o\alpha(x) = \beta(\alpha(x))$$
$$= \beta\left(x^2 - 3x + 7\right)$$
$$= \frac{2-\left(x^2-3x+7\right)}{\left(x^2-3x+7\right)+1}$$
$$= \frac{-x^2+3x-5}{x^2-3x+8}$$